FINITE SOLITARY WAVE IN THE FIELD OF A VORTEX NEAR THE SURFACE OF A HEAVY FLUID

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From the theory of waves of finite amplitude it is known that, with an increase in amplitude the wave becomes steeper and for a certain value of the amplitude a_0 a break appears at its crest. The angle of the tangent at this point, as shown by Stokes, always equals 120°. For larger values of a the wave disintegrates.

It may be expected that the limiting wave will also appear in the field of a vortex located below and close to, the surface of a heavy fluid. As has been shown by Moiseev [1] for the range of Froude numbers just above unity there are in any case two possible solutions for this problem. The existence of the solution which describes the particular flow which degenerates into a plane-parallel flow as the intensity of the vortex approaches zero, was proven by Ter-Krikorov [2]. In [3] the existence of a second solution was established which under identical conditions produces a solitary wave.

In the present paper we shall consider the flow pattern created jointly by the vortex and the fluid flow past it. The free surface of the latter has a singularity at the crest of the wave of the type of a finite solitary wave. It approaches the latter as the intensity of the vortex approaches zero.

We shall investigate the motion of a vortex of intensity Γ under the surface of a heavy fluid of depth H. We shall assume that the vortex moves with a velocity c such that the dimensionless velocity or the Froude number, $F^2 = c^2/gH > 1$. We assume the motion to be steady and to have a potential velocity $w(z) = \phi(x, y) + i\psi(x, y)$.

Let us consider dimensionless variables in the physical (z) plane (Fig. 1,a) of the flow; the problem of the motion of the vortex, situated at the point $z = ia$, will be reduced to the determination in the region D of the analytical function $w(z)$ which has a logarithmic singularity at the point $z = ia$ and the right singularity at the wave crest and which satisfies the boundary and the asymptotic conditions.

$$\frac{1}{2}\left|\frac{dw}{dz}\right|^2 + \nu Y(x) = \text{const}, \qquad \nu = \frac{1}{F^2} \tag{1}$$

$$\psi = 1 \quad \text{on } (L), \qquad \psi = 0 \quad \text{on } (S) \tag{2}$$

$$\lim Y(x) = 1, \qquad \lim \frac{dw}{dz} = 1 \quad \text{for } x \to \infty \tag{3}$$

Let us map the region D in the z plane into a strip of unit width D' in the plane ζ (Fig. 1,b) by the use of the function $\zeta(z) = \xi(x, y) + i\eta(x, y)$, which is analytic in D' except at the wave crest. The point $z = ia$ corresponds to the point $\zeta = i\beta$, the streamlines L and S in the z plane transform into the streamlines L' and S' in the plane ζ.

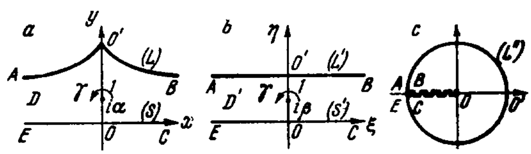

Fig. 1.

The intensity of the vortex does not change as a result of the conformal transformation. The complex potential in the plane ζ has the form

$$w(\zeta) = \zeta + \frac{\gamma}{2\pi i} \ln \frac{\sinh[{}^1/_2\,\pi(\zeta - i\beta)]}{\sinh[{}^1/_2\,\pi(\zeta + i\beta)]}, \qquad \gamma = \frac{\Gamma}{cH} \tag{4}$$

also

$$\operatorname{Im} w(\zeta) = 1 \quad \text{for } \eta = 1, \qquad \operatorname{Im} w(\zeta) = 0 \quad \text{for } \eta = 0$$

The conditions (1), (2) and (3) will assume the form

$$\frac{1}{2} f^2(\xi)\left|\frac{d\zeta}{dz}\right|^2 + \nu Y(\xi, 1) = \text{const} \tag{5}$$

$$f(\xi) = 1 - \frac{\gamma}{2}\,\frac{\sin \pi\beta}{\cosh \pi\xi + \cos \pi\beta}, \qquad \text{const} = \frac{1}{2} + \nu$$

$$\lim Y(\xi, 1) = 1, \qquad \lim \frac{d\zeta}{dz} = 1, \quad \text{for } |\xi| \to \infty, \qquad Y(\xi, 0) = 0$$

In order that $f(\xi) > 0$, it is necessary to fulfil the condition

$$\gamma < 2 \cot({}^1/_2\,\pi\beta)$$

In order to reduce the problem to a nonlinear integral equation let us map the region D' in the ζ plane into the inside of the unit circle in the t plane (Fig. 1,c) by the use of the function

$$\frac{1-t}{1+t}=\cosh\frac{1}{2}\pi\zeta$$

We assume

$$\frac{d\zeta}{dz}=qe^{-i\theta}=e^{-i\omega(\zeta)}, \qquad \omega(\zeta)=\theta(\xi,\eta)+i\tau(\xi,\eta)$$

Then

$$dz=\frac{2i}{\pi}\frac{e^{i\omega(t)}}{\sqrt{t}\,(1+t)}\,dt, \qquad dz=-ds\,e^{i\theta} \quad \text{on } L \tag{6}$$

$$\frac{\partial Y}{ds}=-\sin\theta, \qquad ds=\frac{1}{\pi}\frac{d\sigma}{q\cos{}^1\!/_2\,\sigma}, \qquad \alpha=1-\operatorname{Im}\int_{i\beta}^{i-\infty}e^{i\omega(\zeta)}d\zeta$$

where $e^{i\sigma}=t$ on the circumference L'' in the t plane.

Taking into account (6), we differentiate (5) along the streamline in the z plane, that is along s; we obtain

$$\frac{dq^3}{d\sigma}=\frac{3\nu}{\pi}\frac{\sin\theta(\sigma)\sec{}^1\!/_2\,\sigma}{f^2[\xi(\sigma)]}+\frac{3}{\pi}q^3(\sigma)\sec\frac{\sigma}{2}\frac{f_\xi'[\xi(\sigma)]}{f[\xi(\sigma)]} \tag{7}$$

Integrating (7) along σ, we obtain

$$q^3(\sigma)=\frac{3\nu}{\pi}\int_0^\sigma\frac{\sin\theta(\sigma)\sec{}^1\!/_2\,\sigma}{f^2[\xi(\sigma)]}\,d\sigma+\frac{3}{\pi}\int_0^\sigma q^3(\sigma)\sec\frac{\sigma}{2}\frac{f_\xi'[\xi(\sigma)]}{f[\xi(\sigma)]}\,d\sigma \tag{8}$$

where

$$f[\xi(\sigma)]=1-\frac{\gamma}{4}\frac{\sin\pi\beta}{\tan^2({}^1\!/_2\,\sigma)+\cos^2({}^1\!/_2\,\pi\beta)}, \qquad f_\xi'[\xi(\sigma)]=-\frac{\gamma\pi}{4}\frac{\sin\pi\beta\tan{}^1\!/_2\,\sigma\sec{}^1\!/_2\,\sigma}{(\tan^2({}^1\!/_2\,\sigma)+\cos^2({}^1\!/_2\,\pi\beta))^2}$$

Since $q(\sigma)=1$ for $\sigma=\pi$, then from (8) we have

$$\nu=\frac{\pi}{3}\left(1-\frac{3}{\pi}\int_0^\pi\frac{q^3(\sigma)\sec{}^1\!/_2\,\sigma}{f[\xi(\sigma)]}f_\xi'[\xi(\sigma)]\,d\sigma\right)\left(\int_0^\pi\frac{\sin\theta(\sigma)\sec{}^1\!/_2\,\sigma}{f^2[\xi(\sigma)]}\,d\sigma\right)^{-1} \tag{9}$$

Using Mitchell's method for the solution of the problem of finite waves, it is easy to see that $\omega(t)$ in the region of a singular point has the form [4]

$$\omega(t)=\omega_0(t)+\omega_r(t)$$

where

$$\omega_0(t) = \frac{i}{3}\ln(1-t) - \frac{i}{3}\ln 2, \qquad \omega_r(t) = \sum_{n=0}^{\infty} a_n t^n \tag{10}$$

also the function $\omega_r(t)$ is holomorphic inside the circle and continuous on its boundary. Since we shall investigate a symmetrical wave, it is easy to see that the coefficients a_n are purely imaginary, that is $a_n = ic_n$.

From (10) it is seen that

$$\omega_0(\sigma) = \frac{1}{3}\ln\left|\sin\frac{\sigma}{2}\right| + \left\{\begin{array}{l} \frac{\pi-\sigma}{6}, \ \pi \geqslant \sigma > 0 \\ \frac{-\pi-\sigma}{6}, \ 0 > \sigma > -\pi \end{array}\right\}$$

$$\omega_r(\sigma) = -\sum_{n=0}^{\infty} c_n \sin n\sigma + i \sum_{n=0}^{\infty} c_n \cos n\sigma$$

Then (8) will assume the form

$$\sum_{n=0}^{\infty} c_n \cos n\sigma = \frac{1}{3}\ln\left\{\operatorname{cosec}\frac{\sigma}{2}\left[\frac{3\nu}{\pi}\int_0^{\sigma}\frac{\sec{}^1\!/_2\sigma}{f^2[\xi(\sigma)]}\left[\sin\left(\frac{\pi-\sigma}{6} - \sum_{n=0}^{\infty} c_n \sin n\sigma\right)\right]d\sigma + \right.\right.$$
$$\left. + \frac{3}{\pi}\int_0^{\sigma}\frac{\tan{}^1\!/_2\sigma}{f[\varepsilon(\sigma)]}\exp\left(3\sum_{n=0}^{\infty} c_n \cos n\sigma\right) f_{\xi}'[\xi(\sigma)]\,d\sigma\right]\Bigg\} \tag{11}$$

where ν is the function given by (9). Also it follows from (11) for $\sigma = 0$ that

$$\nu = \frac{\pi}{3} f^2[\xi(0)]\exp\left(3\sum_{n=0}^{\infty} c_n\right)$$

The limiting value of the amplitude at which a finite wave appears is $a_0 = 1/2\ \nu$.

The value P of the lifting force is given by

$$P = -\frac{\pi}{8}\rho c\Gamma C_y, \qquad C_y = \left(\frac{\cos{}^1\!/_2\pi\beta}{1+\cos{}^1\!/_2\pi\beta}\right)^{1/3}\exp\sum_{n=0}^{\infty} c_n \tan^{2n}\frac{\pi\beta}{4} \tag{12}$$

Equation (11) was solved numerically. The first twelve coefficients c_n were determined by the method of iteration at specified points. The problem was computed on the machine "Strela". In Figures 2 and 3 are shown the curves of ν and c_y as functions of γ and β. As seen from Figure 2 there exists a unique value ν_0 (or c_0) for each value of γ and β at which there appears a finite wave, and for $\nu < \nu_0$ (or $c > c_0$) disintegration of the wave takes place.

For constant values of ν and β the curves of Figure 2 yield a finite value γ, for which a finite wave appears. In Figure 3 are given the values of the lift coefficient C_y for the critical values γ, β, ν (or c). Analysis of the solution shows that a finite symmetrical wave is possible only for $\gamma < 0$ and for small positive values γ.

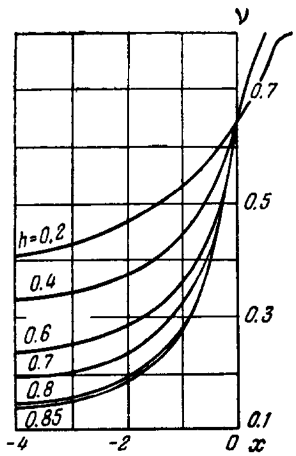

Fig. 2.

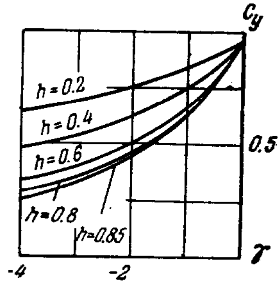

Fig. 3.

Note. We observe the fact that in the case of parallel flow past a vortex which transforms into a plane-parallel flow for γ, (or Γ) $\to$ 0, a finite Stokes wave does not exist for the following reason:

Since the stream for γ (or Γ) $\to$ 0 transforms into a plane-parallel flow, the function $\zeta = \zeta\,(z,\ \gamma)$, which maps the physical plane of the flow upon a strip of unit width, will assume the form

$$\zeta\,(z, \gamma) = z + \gamma^k \zeta_1\,(z, \gamma) \qquad (k > 0), \qquad \zeta\,(z, 0) \neq 0$$

where γ is a real quantity. We have

$$\left|\frac{d\zeta}{dz}\right|^2 = 1 + 2\gamma^k q_1 \cos\theta_1 + \gamma^{2k} q_1^2 \qquad \left(\frac{d\zeta_1}{dz} = q_1 e^{-i\theta_1}\right) \tag{13}$$

Since in the case of Stokes' wave $|d\zeta/dz|^2 = 0$ at the wave crest, (13) yields

$$1 + 2\gamma^k q_{10} \cos\theta_{10} + \gamma^{2k}\ q_{10}^2 = 0, \qquad q_{10} = q_1\,(0,1), \qquad \theta_{10} = \theta_1\,(0,1)$$

or

$$\gamma^k = -\ q_{10} \cos\theta_{10} + i q_{10} \sin\theta_{10}$$

that is, a Stokes wave may exist only for complex values of γ (or Γ), which is in contradiction with the condition that γ is a real quantity.

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